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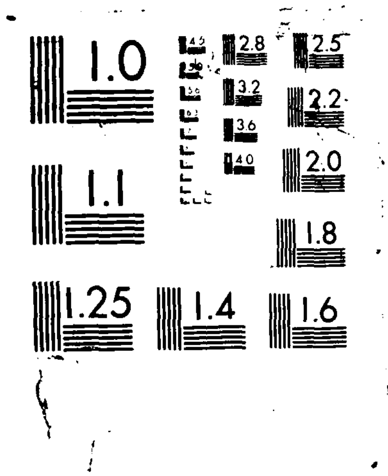
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SOLVING SINGULAR SYSTEMS USING ORTHOGONAL FUNCTIONS

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Abstract

Orthogonal functions, and in particular, Walsh functions, have been advocated in the literature as a method of approximating the solutions of singular systems $Ex' = Fx + Bu$ of index k. This paper gives the first analysis of the accuracy of these approximations. For Walsh functions, divergence is shown for $k \geq 3$ and convergence for $k = 0, 1$. The index two case is also analyzed.

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1 Introduction

The singular control system

$$Ex'(t) = Fx(t) + Bu(t), \quad x(t_0) = x_0 \tag{1}$$

with E, F, B constant matrices and E singular, has been extensively studied [2],[3],[8]. In [11] it was suggested that (1) could be solved using orthogonal functions. This was discussed further in [4],[10],[9]. These papers considered Walsh functions because of their simple structure and the ease of approximating coefficients. While these papers showed that one could solve the resulting algebraic equations for the coefficients of an approximation, none of them actually examined how good these approximations were. In this paper we shall give the first discussion of the convergence of the Walsh approximations for singular systems. It will be shown that in many cases the approximations actually diverge from the true solutions as more terms are used in the approximation.

2 Orthogonal Approximations

Suppose that E, F are $n \times n$ and that (1) is solvable. That is, $\lambda E + F$ is a regular pencil so that $\det(\lambda E + F) \not\equiv 0$ and (1) has a solution for every sufficiently smooth u and for consistent x_0 [2]. We consider real E, F, B, x, u but the complex case is similar.

To simplify notation,. In order to explain our analysis we need to introduce some notation and review some terminology from the theory of orthogonal functions. Let $\mathcal{L}^2$ be the space of all square integrable lebesque measurable functions on $[0,1]$. $\mathcal{L}^2$ is a Hilbert space with inner product,

$$< f, g > = \int_0^1 f(t)g(t)dt \tag{2}$$

and associated norm

$$\|f\| = \left(\int_0^1 f(t)^2 dt \right)^{1/2} \tag{3}$$

A vector valued function will be said to be in $\mathcal{L}^2$ if each coordinate is. Let

$\{\psi_i(t)\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{L}^2$. Then if $f \in \mathcal{L}^2$, we have

$$f(t) = \sum_{i=1}^{\infty} f_i \psi_i(t) \tag{4}$$

where the f_i are the fourier coefficients of f with respect to the basis $\{\psi_i\}$. The series (4) converges in the norm (3). For a given orthonormal basis $\{\psi_i\}$, let $\Psi_m = [\psi_1, \dots, \psi_m]^T$ and define the projection onto the span of $\{\psi_1, \dots, \psi_m\}$ by

$$\mathcal{P}_m(f) = \sum_{i=1}^{m} f_i \psi_i$$

Let the coefficients of this projection be given by the operator

$$\mathcal{C}_m(f) = [f_1, \dots, f_m]$$

For a vector $a = [a_1, \dots, a_m]$, define

$$\mathcal{F}_m(a) = \sum_{i=1}^{m} a_i \psi_i$$

If f is vector valued, then $\mathcal{C}_m(f)$ is a matrix whose ij-th entry is the j-th fourier coefficient of the i-th element of f. Similarly, the a_i in the definition of $\mathcal{C}_m$ can be vectors. Note that $\mathcal{P}_m(f) = \mathcal{C}_m(f)\Psi_m$. Finally, define the $m \times m$ matrix P_m by

$$\mathcal{C}_m\left(\int_0^t \Psi_m(s)ds\right) = P_m$$

Now we can consider the singular system (1). Fix m and take $X = \mathcal{C}_m(x), U = \mathcal{C}_m(u), Q = \mathcal{C}_m(x_0)$ where x_0 is considered a constant function. Integrating (1) gives

$$Ex(t) - Ex_0 = F\int_0^t x(s)ds - B\int_0^t u(s)ds \tag{5}$$

Using the approximations $x \approx X\Psi, u \approx U\Psi$, and $x_0 = Q\Psi$ in (5) gives

$$EX\Psi - EQ\Psi = FX\int_0^t \Psi(s)ds - BU\int_0^t \Psi(s)ds \tag{6}$$

where the m subscript has been dropped. Taking $\mathcal{C}_m$ of both sides of (6) and letting $P = P_m$ yields the algebraic equation

$$EX - EQ = FXP - BUP \tag{7}$$

In the method of orthogonal functions, (7) is solved for X given E, F, P, U, B. The most discussed orthonormal basis to date have been the Walsh functions [5], [9],[10],[13]. If $m = 2^j$, however, the span of the first m Walsh functions is identical to the linear span of the m block-pulse functions $\{\hat{\phi}_1, \ldots, \hat{\phi}_m\}$ where $\hat{\phi}(t)$ is 1 if $\frac{i-1}{m} \leq t \leq \frac{i}{m}$ and 0 otherwise. The set $\{\hat{\phi}\}_{i=1}^m$ is orthogonal and can be normalized by multiplying by $\sqrt{m}$. Let $\phi_i = \sqrt{m}\hat{\phi}_i$. Notice that the ϕ_i are not an orthonormal basis for $\mathcal{L}^2$. Rather for $m = 2^j$, the $\{\phi_i, \ldots, \phi_m\}$ is an orthonormal basis for the span of the first m Walsh functions. Thus we get approximations with the same error using either set of m functions and the linear algebra problem (7) has the same numerical conditioning in both cases. That (7) has a solution is shown in [9]. We are interested in the accuracy of the approximation to the solution x given by the solution X of (7).

Using the standard structure theory for matrix pencils [2],[8], we can transform (1) by constant coordinate changes into

$$z_1' = Cz_1 + B_1 u \tag{8}$$
$$Nz_2' = z_2 + B_2 u \tag{9}$$

where N is nilpotent of index k. That is, $N^k = 0, N^{k-1} \neq 0$. Similarily, (7) will decouple into two equations, one for the coefficients of z_1 and one for the coefficients of z_2. Let $h = 1/m$. It is known that the Walsh functions will give an $O(h)$ approximation for (8). We consider then only (9). Additional coordinate changes on (9) will put N into Jordan form and decouple (7) so that the subsystems may be considered separately. Thus there is no loss of generality in assuming that (1) is in the form of (9) where N is an elementary Jordan block of index k. We now carefully consider the index 3 case.

Example 1. Consider (1) where

$$E = N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad F = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 7 \\ -1 \\ 1 \end{bmatrix}, \quad u = 1 + 4t - 3t^2 - 2e^t$$

This system has the solution

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 + 2t + 3t^2 + 6e^t \\ 2t + 3t^2 + 4e^t - 5 \\ 3t^2 + 2e^t - 4t - 1 \end{bmatrix}$$

The resulting equation (7) was solved using the numerically robust Bartels-Stewart algorithm [1],[6],[7]. Table 1 gives the $\mathcal{L}^2$ error, $||x - \mathcal{F}_m(X)||$, for several values of m.

m	x_1	x_2	x_3
4	12.67	0.954	0.259
8	24.79	0.478	0.131
16	49.34	0.239	0.065
32	98.57	0.119	0.032
64	197.09	0.059	0.016

Table 1. $\mathcal{L}^2$ error for Example 1

From this table we observe what appears to be $O(h)$ convergence in x_2 and x_3 and $O(\frac{1}{h})$ divergence in x_1. The convergence in x_2 was surprising. We had expected the error in approximating x_2 to be $O(1)$. To understand the convergence for the x_2 variable and to show that the observed behavior of this example reflects the general case, we shall consider this example more carefully. For the block pulse functions,

$$P_m = \frac{1}{2m+1} \begin{bmatrix} 1 & 2 & 2 & * & 2 \\ 0 & 1 & 2 & * & * \\ & \cdot & * & * & * \\ & \cdot & \cdot & 1 & 2 \\ 0 & \cdot & \cdot & 0 & 1 \end{bmatrix} = \frac{h}{2}\hat{P}$$

and

$$\hat{P}^{-1} = \begin{bmatrix} 1 & -2 & 2 & * & \pm 2 \\ 0 & 1 & -2 & * & * \\ \cdot & \cdot & * & * & 2 \\ \cdot & \cdot & \cdot & 1 & -2 \\ 0 & \cdot & \cdot & 0 & 1 \end{bmatrix}$$

Substituting the values of E, F, P, U, B, Q from Example 1 into (7) gives

$$\begin{aligned} &\begin{bmatrix} x_{21} & \cdots & x_{2m} \\ x_{31} & \cdots & x_{3m} \\ 0 & \cdots & 0 \end{bmatrix} - \begin{bmatrix} x_{11} & \cdots & x_{1m} \\ x_{21} & \cdots & x_{2m} \\ x_{31} & \cdots & x_{3m} \end{bmatrix} \frac{h}{2}\hat{P} \\ &= \begin{bmatrix} u_{11} & \cdots & u_{1m} \\ u_{11} & \cdots & u_{1m} \\ u_{11} & \cdots & u_{1m} \end{bmatrix} \frac{h}{2}\hat{P} + \begin{bmatrix} q_{21} & \cdots & q_{2m} \\ q_{31} & \cdots & q_{3m} \\ 0 & \cdots & 0 \end{bmatrix} \end{aligned} \tag{10}$$

From the third row of (10) we have

$$-[x_{31}, \ldots, x_{3m}] = [u_{11}, \ldots, u_{1m}]$$

But the exact solution is $x_3 = -u$. Thus the algebraic variable x_3 is approximated to the same accuracy as u was approximated which is $O(h)$. Now for x_2 we have

$$\begin{aligned} [x_{21}, \ldots, x_{2m}] &= -[u_{11}, \ldots, u_{1m}] \\ &\quad + ([x_{31}, \ldots, x_{3m}] - [q_{31}, \ldots, q_{3m}]) \left(\frac{h}{2}\hat{P}\right)^{-1} \end{aligned} \tag{11}$$

The actual solution is $x_2 = x_3' - u = -u' - u$. Thus

$$([x_{31}, \ldots, x_{3m}] - [q_{31}, \ldots, q_{3m}]) \left(\frac{h}{2}\hat{P}\right)^{-1} \tag{12}$$

must be an approximation for x_3'. To see why this is $O(h)$ even though the Euclidian operator norm of $\frac{2}{h}\hat{P}^{-1}$ is $O(\frac{1}{h^2})$, let r be a function of t and $g(t) = \int_0^t r(s)ds$. Let $G = \mathcal{C}_m(g)$, $R = \mathcal{C}_m(r)$ with respect to the block pulse functions. Then

$$\mathcal{C}_m(g) = \mathcal{C}_m\left(\int_0^t r(s)ds\right) = \mathcal{C}_m\left(\int_0^t \mathcal{P}_m(r)ds\right) + \mathcal{C}_m\left(\int_0^t (I - \mathcal{P}_m)(r)ds\right)$$

or

$$G = RP + \epsilon_m$$

and finally

$$GP^{-1} = R + \epsilon_m P^{-1}$$

The term $\epsilon_m P^{-1}$ is the error in using GP^{-1} as an estimate for R. In our example we have that $r = x_3'$, $g = x_3 - x_3(0)$ and $G = [x_{31}, \ldots, x_{3m}] - [q_{31}, \ldots, q_{3m}]$. It is not clear that $||\mathcal{F}_m(\epsilon_m P^{-1})||$ is $O(h)$. Suppose that r has slowly varying slope. The interval $[0,1]$ can always be broken into subintervals on which this is true. Then a straightforward, but tedious calculation, shows that ϵ_m looks like $h^{5/2} M \pi_m$ where π_m is a vector of ones and M is independent of m and hence $||\mathcal{F}_m(\pi_m)|| = \sqrt{m} = O(h^{-1/2})$. (Actually the entries of ϵ_m are between two vectors in this form.) But $\delta_m = \pi \hat{P}^{-1}$ is a vector of ones and minus ones. Thus

$$||\mathcal{F}_m(\epsilon_m P^{-1})|| = ||\mathcal{F}_m(h^{5/2} M \pi_m \frac{2}{h} \hat{P}^{-1})|| = 2h^{3/2} M ||\mathcal{F}_m(\delta)|| = 2Mh$$

as was observed for x_2. However, this error term gets multiplied again by P^{-1} in the computation of the estimate for x_1 and $\delta \hat{P}^{-1}$ looks like $[1, 3, 5, \ldots]$ which are the normalized block pulse coefficients of a function of norm $O(\frac{1}{h^{3/2}})$. Thus for x_1 we have

$$\begin{aligned} ||\mathcal{F}_m(\epsilon_m P^{-1} P^{-1})|| &= ||\mathcal{F}_m(2h^{3/2} M \delta P^{-1})|| \\ &= 2Mh^{1/2} ||\mathcal{F}_m(\delta \hat{P}^{-1})|| = 2Mh^{1/2} O(\frac{1}{h^{3/2}}) = O(\frac{1}{h}) \end{aligned}$$

as observed.

3 Discussion

As noted earlier the example studied is typical of all systems of index $k \leq 3$. Also, any system index higher then 3 must contain a subsystem of index 3 of the form of our example. Thus we can make the following conclusions about the use of Walsh functions on singular systems.

1. The orthogonal function method using Walsh functions will give an $O(h)$ approximation for systems (1) of index zero or one.

2. The method will diverge for singular systems of index greater then two.

3. For index two systems, we can expect $O(h)$ convergence. However, this depends on algebraic cancellation and can be expected to be numerically sensitive. Also, convergence can be reduced by controls u with rapidly changing derivatives.

The difficulty in using Walsh functions arises because of the integral approximation. If our orthogonal basis had the property that

$$(I - \mathcal{P}_m)\left(\int_0^t \Psi_m(s)ds\right) = 0$$

then there would not be this difficulty. One example of such a basis is $\{\cos x, \sin x, \cos 2x, \sin 2x, \ldots\}$ on $[-\pi, \pi]$ and m an even integer. However, in this case the coefficients are much more difficult to compute.

Even if (1) is a system on which the method of orthogonal functions using Walsh functions may work, this method may not be the method of choice unless there is some particlar need or reason for using Walsh functions. Large values of m must be used to obtain even modest accuracy. There are several alternatives that can provide higher accuracy at substantially less cost. Among these are computation of part of the matrix pencil [15],[7], backward differentiation formulas [14], and implicit Runge Kuttas [12].

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